A Gentle Introduction to the Dirac Delta

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Consider the following position vs time profile:

$$x = f_+(t, t_0) = \begin{cases} 0 & t \leq t_0 \\ t & t > t_0 \end{cases} \tag{1}$$

Make the following observation

$$x = f_+(t, t_0) = \begin{cases} 0 & t \leq t_0 \\ \sqrt{t^2} & t > t_0 \end{cases} \tag{2}$$

can be use to represent the same profile.
Now differentiate the first form of the profile.

$$\frac{df_+}{dt} = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \end{cases} \tag{3}$$

Differentiate the second form the same profile.

$$\frac{df_+}{dt} = \begin{cases} 0 & t < t_0 \\ \frac{t}{\sqrt{t^2}} & t > t_0 \end{cases} \tag{4}$$

This can be thought of as the unit step (impulse), Heaviside function. It is to be noted that this function is undefined at $t = t_0$.

Performing another differentiation on form 1 and form 2 results in.

$$\frac{d^2 f_+}{dt^2} = \begin{cases} 0 & t < t_0 \\ 0 & t > t_0 \end{cases} \tag{5}$$

$$\frac{d^2 f_+}{dt^2} = \begin{cases} 0 & t < t_0 \\ \frac{1}{\sqrt{t^2}}_b - \frac{1}{\sqrt{t^2}}_t & t > t_0 \end{cases} \tag{6}$$

With this information, an operational definition of the δ_+ function.

$$\delta_+(t_0 - t) = 0 t \neq t_0 \tag{7}$$

$$\int \delta_+(t_0 - t) d(t) = 1 \tag{8}$$

This is a very strange "function", zero everywhere except at x=0 where it is undefined, and the definite integral is 1.

$$\int f(t)\delta(t_0 - t)dt = \int (f(t_0) + f'(t) * (t_0 - t) + f''(t) * (t_0 - t)^2 + ...)\delta(t_0 - t)dt \tag{15}$$

Noting that

$$f(t_0) \int \delta(t_0 - t)dt = f(t_0) \tag{16}$$

At this point no further differentiation is possible if expressed in form 1, but form 2 allows for further differentiation, in particular taking one more derivative on expression 6 yields.

$$\frac{d\delta_+}{dt^2} = -\frac{1}{t}\delta(t) \tag{9}$$

with this relationship all additional derivatives can be found directly

$$\frac{d^n \delta_+}{dt^n} = -1^n \frac{1}{t^n}\delta(t) \tag{10}$$

A similar analysis along the same line as above can be repeated with a position vs time profile of

$$f_-(t, t_0) - = \begin{cases} -t & t \leq t_0 \\ 0 & t > t_0 \end{cases} \tag{11}$$

Define $\delta = \frac{1}{2}(\delta_- + \delta_+)$ corresponding to position vs time profile of

$$|t| = \begin{cases} -t & t \leq t_0 \\ t & t > t_0 \end{cases} \tag{12}$$

resulting in the traditional delta Dirac function used in analysis. Note that δ is a symmetric "function", $\delta(x) = \delta(-x)$.

The integral properties of the *delta* can be examined throught the use of the integration by parts, , and it's differental properties. Start with

$$\int \delta(t)d(t) = (t\delta(t)) + \int \delta(t)d(t) \tag{13}$$

This tells us that $(t)\delta(t) = 0$. Meaing that $t = 0$, thus $t\delta(t_0 - t)dt = (t_0)\delta(t_0 - t)$. By similar arguments

$$(t_0 - t)^n \delta(t_0 - t) = (t_0)^n \delta(t_0 - t) = 0 \tag{14}$$

Consider any well behaved function, one which a taylor expansion is possible.

$$\int f(t)\delta(t_0 - t)dt = f(t_0) \tag{17}$$